Journal of Engineering Mathematics **38:** 391–402, 2000.
© 2000 *Kluwer Academic Publishers. Printed in the Netherlands.*

Nonlinear capillary-gravity waves under an edge condition

C. C. CHANG and M. C. SHEN
Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

Received 6 January 1999; Accepted in revised form 20 March 2000

Abstract. The objective of this paper is to investigate the forced motion of nonlinear capillary-gravity waves in a waterfilled circular basin by a harmonic vibration applied to its side wall under Evans's or Hocking's edge condition at a contact line. The forcing frequency is near a resonance frequency under the classical edge condition of the basin. Two complex-amplitude equations for the excited eigenmode at the resonance frequency corresponding to these two edge conditions are derived. The solutions to these equations display quite different behavior and an edge condition indeed has a great influence on the excited surface waves.

Key words: capillary-gravity waves, edge conditions, resonance frequency, eigenmode, amplitude equations

1. Introduction

The problem of forced capillary-gravity waves generated by an external harmonic vibration applied to part of the container boundary has been studied in the past. The experimental work on a water-filled circular basin was reported by Wang *et al.* [1], and later by Hsieh and Denissenko [2]. The generation of capillary-gravity waves in a circular cylinder with a fixed contact line was discussed by Miles and Henderson [3], among others. Shen, Sun and Hsieh [4] dealt with the linear problem of forced capillary-gravity waves generated in a water-filled circular basin by a prescribed harmonic forcing to the side wall. At a contact line where the side wall and the free surface intersect, they prescribed Evans's edge condition [5] that the slope of the free surface in the plane normal to the side wall at the contact line (normal surface slope) is proportional to the prescribed wall motion of the side wall. An exact solution was obtained by means of the Green's function method. In addition, Shen and Yeh [6] presented an exact solution derived by the same method for the linear problem under Hocking's edge condition [7] at the contact line, that is, the time derivative of the free surface is proportional to the normal surface slope. The eigenvalue problem under Hocking's edge condition was studied by Miles [8] by a different approach. He obtained the solution given in [4] and an approximate solution [9] for the same linear problem studied in [6].

The objectives of this paper are to develop a weakly nonlinear theory by an asymptotic approach for an excited capillary-gravity wave in a water-filled circular basin under Evans's or Hocking's edge condition at a contact line and to compare the results under these two edge conditions. We assume that the forcing frequency is near a resonance frequency under the classical edge condition that the normal surface slope at the contact line vanishes. By means of a two-timing asymptotic expansion and a solvability condition for the equations of the third order approximations in the asymptotic expansion, the amplitude equations of the excited surface waves at the resonance frequency are derived. A weakly nonlinear theory for excited surface waves in a water-filled circular basin without surface tension was developed by Sun, Shen and Hsieh [10], and the results obtained in [10] has characteristics similar to

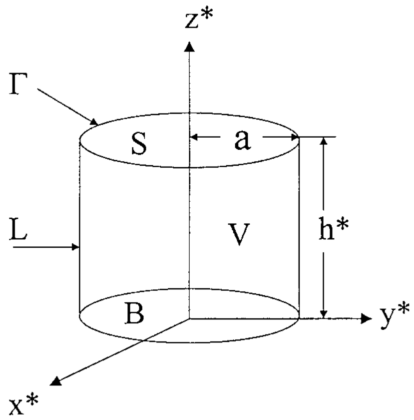

Figure 1. The circular basin.

those under Evans's condition obtained here. We also remark that in order to carry out an asymptotic investigation for our objectives certain parameters in the edge conditions have to be assumed small. Under Evans's edge condition, the amplitude equation can be reduced to a two-dimensional Hamiltonian system. The solutions are bounded and the orbits are closed. However, under Hocking's edge condition, there is no closed orbit in the phase plane of the amplitude function. Furthermore, the absolute value of a solution of the amplitude equation for all orbits except the heteroclinic ones goes to infinity as time goes to infinity. Our work thus indicates that an edge condition indeed has a great effect on the behavior of the excited surface waves.

In Section 2, the problem is formulated with Evans's or Hocking's edge condition prescribed at a contact line. A two-timing method is then used to derive a complex-amplitude equation under each edge condition in Sections 3 and 4. The behavior of solutions of the amplitude equations is discussed in Section 5, and some numerical results are also presented there. The tedious coefficients in the derivation of amplitude equations, which are obtained by symbolic manipulations via Mathematica, will not be presented in the paper, but are available on request.

2. Formulation

We consider a circular basin filled with water and an external harmonic forcing applied to the side wall of the basin. Let V be the open domain occupied by the water, S be the free surface denoted by $z^* = \eta^*(r^*, \theta, t^*)$ where r^*, θ, z^* are the cylindrical coordinates and t^* is the time, L be the side wall $r^* = a + F^*(\theta, z^*, t^*)$ where F^* is a prescribed forcing function, Γ be the contact line, and B be the bottom of the basin on $z^* = -h^*$ (Figure 1). Assume that the water motion is irrotational, and the governing equation and boundary conditions with surface tension in terms of a potential function ϕ^* are

$$\phi^*_{r^*r^*} + (r^*)^{-1}\phi^*_{r^*} + (r^*)^{-2}\phi^*_{\theta\theta} + \phi^*_{z^*z^*} = 0 \quad \text{in } V, \tag{1}$$

at the free surface S

$$\eta^*_{t^*} + (r^*)^{-2}\eta^*_{\theta}\phi^*_{\theta} + \eta^*_{r^*}\phi^*_{r^*} - \phi^*_{z^*} = 0, \tag{2}$$

$$\phi^*_{t^*} + (1/2)((\phi^*_{r^*})^2 + (r^*)^{-2}(\phi^*_\theta)^*)^2 + (\phi^*_{z^*})^2) + g\eta^* = T^*\nabla^* \cdot (\nabla^*\eta^*/(1 + |\nabla^*\eta^*|^2)^{1/2}), \tag{3}$$

at the side wall L

$$F^*_{t^*} - \phi^*_{t^*} + (r^*)^{-2}F^*_\theta\phi^*_\theta + F^*_{z^*}\phi^*_{z^*} = 0, \tag{4}$$

at the bottom B

$$\phi^*_{z^*} = 0, \tag{5}$$

where g is the constant gravitational acceleration, T^* is the constant surface tension coefficient, and ∇^* is the gradient in the r^*, θ, z^*-coordinates.

In the following, Evans's and Hocking's edge conditions are introduced at a contact line Γ. At Γ, under Evans's edge condition,

$$\eta^*_{r^*} = \tilde{K}(\theta, t^*), \tag{6}$$

and under Hocking's edge condition,

$$\eta^*_{t^*} = \lambda^*\eta^*_{r^*}, \tag{7}$$

where $\tilde{K}$ is a given smooth function of θ and t^*, and λ^* is a positive constant.

To nondimensionalize (1) to (7), we introduce the following nondimensional variables

$$z = z^*/a,\ r = r^*/a,\ h = h^*/a,\ \eta = \eta^*/a,\ \tilde{F} = F^*/a,\ t = (g/a)^{1/2}t^*,$$
$$\phi = \phi^*/(a(ga)^{1/2}),\ T = T^*/(ga^2),\ \tilde{\lambda} = \lambda^*/(ga)^{1/2},$$

and in terms of them, (1) to (7) become

$$\Delta_3\phi = 0 \quad \text{in } V, \tag{8}$$

at $z = \eta(r, \theta, t)$:

$$\eta_t + r^{-2}\eta_\theta\phi_\theta + \eta_r\phi_r - \phi_z = 0, \tag{9}$$

$$\phi_t + (1/2)(\phi_r^2 + r^{-2}\phi_\theta^2 + \phi_z^2) + \eta = T\nabla \cdot (\nabla_\eta(1 + |\nabla\eta|^2)^{1/2}), \tag{10}$$

at $r = 1 + \tilde{F}(z, \theta, t)$:

$$\tilde{F}_t - \phi_r + (1 + \tilde{F})^{-2}\tilde{F}_\theta\phi_\theta + \tilde{F}_z\phi_z = 0, \tag{11}$$

at $z = -h$:

$$\phi_z = 0, \tag{12}$$

and at $z = \eta(r, \theta, t)$, $r = 1 + \tilde{F}(z, \theta, t)$, under Evans's edge condition,

$$\eta_r = \tilde{K}(\theta, t/(g/a)^{1/2}), \tag{13}$$

and under Hocking's edge condition,

$$\eta_r = \tilde{\delta}\eta_t, \tag{14}$$

where $\Delta_3 = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\theta^2 + \partial^2/\partial z^2$, ∇ is the gradient in the $r - \theta - z$ coordinates, and $\tilde{\delta} = \tilde{\lambda}^{-1}$.

We study equations (8) to (14) by means of a two timing asymptotic expansion. First we define the slow time τ by $\tau = \varepsilon^2 t$ where $\varepsilon > 0$ is a small parameter, and assume

$$\tilde{F} = \varepsilon^3 F(z, \theta, t, \tau) = \varepsilon^3 (\mathrm{i}\omega)^{-1} f(z) \cos m\theta (\mathrm{e}^{\mathrm{i}(\omega t + \lambda\tau + \delta)} - \mathrm{e}^{-\mathrm{i}(\omega t + \lambda\tau + \delta)}), \tag{15}$$

$$\tilde{K} = \varepsilon^3 K(\theta, t, \tau) = \varepsilon^3 (\mathrm{i}\omega)^{-1} \alpha \cos m\theta (\mathrm{e}^{\mathrm{i}(\omega t + \lambda\tau + \delta)} - \mathrm{e}^{-\mathrm{i}(\omega t + \lambda\tau + \delta)}), \tag{16}$$

$$\tilde{\delta} = \varepsilon^2 \hat{\delta}, \tag{17}$$

$$\phi = \phi(r, \theta, z, t, \tau) = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \cdots, \tag{18}$$

$$\eta = \eta(r, \theta, t, \tau) = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + \cdots, \tag{19}$$

where ϕ_i and η_i, $i = 1, 2, 3, \ldots$, are independent of ε. We note here that the amplitude of the harmonic forcing function $\tilde{F}$ is assumed small and consequently the amplitude of $\tilde{K}$ in the Evans's condition is of the same order as $\tilde{F}$. Furthermore, we assume the nondimensional parameter $\tilde{\delta} = (ga)^{1/2}/\lambda^*$ in the Hocking's condition is also small. The precise orderings of these parameters are specified so that the amplitude of the eigenmode corresponding to a resonance frequency under the classical edge condition can be determined by the equations for ϕ_3.

Substitution of (15) to (19) in (8) to (14) will yield a sequence of equations for the successive approximations.

3. First- and second-order approximations

For both edge conditions, the equations for the first-order approximations are the same as follows:

$$\Delta_3\phi_1 = 0 \quad \text{in } 0 \le r < 1, -h < z < 0, \tag{20}$$

$$\phi_{1z} - \eta_{1t} = 0, \quad \phi_{1t} + \eta_1 = T\Delta_2\eta_1 \quad \text{at } z = 0, \tag{21}$$

$$\phi_{1r} = 0 \quad \text{at } r = 1, \tag{22}$$

$$\phi_{1z} = 0 \quad \text{at } z = -h, \tag{23}$$

$$\eta_{1r} = 0 \quad \text{at } r = 1 \text{ and } z = 0, \tag{24}$$

where $\Delta_2 = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\theta^2$.

It follows from (21) and (24) that

$$\phi_{1tt} + \phi_{1z} = T\Delta_2\phi_{1z} \quad \text{at } z = 0, \tag{25}$$

$$\phi_{1rz} = 0 \quad \text{at } r = 1, z = 0. \tag{26}$$

Assume that the solutions ϕ_1 and η_1 of (20) to (26) have the form of

$$\phi_1 = (\varphi_1(r, z, \tau)\mathrm{e}^{\mathrm{i}\Omega t} + \overline{\varphi_1}(r, z, \tau)\mathrm{e}^{-\mathrm{i}\Omega t}) \cos m\theta,$$

$$\eta_1 = (\hat{\eta}_1(r, z, \tau)\mathrm{e}^{\mathrm{i}\Omega t} + \overline{\hat{\eta}_1}(r, z, \tau)\mathrm{e}^{-\mathrm{i}\Omega t}) \cos m\theta,$$

Then φ_1 satisfies

$$\varphi_{1rr} + r^{-1}\varphi_{1r} - m^2 r^{-2}\varphi_1 + \varphi_{1zz} = 0 \quad \text{in } 0 \le r < 1, -h < z < 0, \tag{27}$$

$$\varphi_{1z} - \Omega^2\varphi_1 = T(\varphi_{1rrz} + r^{-1}\varphi_{1rz} - m^2 r^{-2}\varphi_{1z}) \quad \text{at } z = 0, \tag{28}$$

$$\varphi_{1r} = 0 \quad \text{at } r = 1, \qquad \varphi_{1z} = 0 \quad \text{at } z = -h, \tag{29}$$

$$\varphi_{1rz} \quad \text{at } r = 1, z = 0, \tag{30}$$

and $\hat{\eta}_1$ satisfies

$$\hat{\eta}_1 = (\mathrm{i}\Omega)^{-1}\varphi_{1z} \quad \text{at } z = 0. \tag{31}$$

Equations (27) to (30) can be solved by separation of variables, and we obtain

$$\varphi_1 = p(\tau) J_m(\sigma_{mj} r) \cosh \sigma_{mj}(h + z),$$

where $p(\tau)$ is a complex function of τ, σ_{mj} are the positive roots of $J'_m(\sigma) = 0$ and $J_m(r)$ is the mth order Bessel function of the first kind. By (27), (28) implies

$$\varphi_{1z} - \Omega^2\varphi_1 = -T\varphi_{1zzz} \quad \text{at } z = 0,$$

and it follows that

$$\Omega^2 = \Omega^2_{mj} = \sigma_{mj}(1 + T\sigma^2_{mj}) \tanh(\sigma_{mj} h), \tag{32}$$

and Ω_{mj} are called the resonance frequencies of the water-filled basin under the classical edge condition, $\eta_r = 0$ at Γ. In what follows, we shall just refer to them as the resonance frequencies of the basin.

Now we assume ω is one of the resonance frequencies Ω_{mn} of the excited wave where n is a positive integer, that is, $\omega = \Omega_{mn}$, and denote Ω_{mn} and σ_{mn} by Ω and σ, respectively, hereafter. Thus

$$\phi_1 = J_m(\sigma r) \cosh \sigma(h + z) \cos m\theta (p(\tau)\mathrm{e}^{\mathrm{i}\Omega t} - \overline{p}(\tau)\mathrm{e}^{-\mathrm{i}\Omega t}), \tag{33}$$

and by (31),

$$\eta_1 = -\mathrm{i}\Omega^{-1}\sigma \sinh(\sigma h) J_m(\sigma r) \cos m\theta (p(\tau)\mathrm{e}^{\mathrm{i}\Omega t} - \overline{p}(\tau)\mathrm{e}^{-\mathrm{i}\Omega t}). \tag{34}$$

Here $p(\tau)$ is called the complex-amplitude function of the excited wave and will be determined by the equations for the third-order approximations.

For both edge conditions, the equations for the second-order approximations are also the same as follows:

$$\Delta_3\phi_2 = 0 \quad \text{in } 0 \le r < 1, -h < z < 0, \tag{35}$$

at $z = 0$,

$$\phi_{2z} - \eta_{2t} = -\phi_{1zz}\eta_1 + \eta_{1r}\phi_{1r} + r^{-2}\eta_{1\theta}\phi_{1\theta} = H_1(\eta_1, \phi_1), \tag{36}$$

$$\begin{aligned} \phi_{2t} + \eta_2 &= -\phi_{1tz}\eta_1 - (1/2)(\phi^2_{1r} + r^{-2}\phi^2_{1\theta} + \phi^2_{1z}) + T(\eta_{2rr} + r^{-2}\eta_{2\theta\theta} + r^{-1}\eta_{2r}) \\ &= H_2(\eta_1, \phi_1) + T\Delta_2\eta_2, \end{aligned} \tag{37}$$

$$\phi_{2r} = 0 \quad \text{at } r = 1, \quad \phi_{2z} = 0 \quad \text{at } z = -h \tag{38}$$

$$\eta_{2r} = 0 \quad \text{at } r = 1, z = 0. \tag{39}$$

Elimination of η_2 in (36) and (37) yields, by (33) and (34), at $z = 0$,

$$\begin{aligned}\phi_{2tt} + \phi_{2z} - T\Delta_2\phi_{2z} &= H_1 + H_{2t} - T\Delta_2 h_1 = H_3 \\ &= \mathrm{i}(W_1(r) + W_2(r)\cos 2m\theta)(p^2\mathrm{e}^{\mathrm{i}\Omega t} - \overline{p}^2\mathrm{e}^{-\mathrm{i}\Omega t}),\end{aligned} \tag{40}$$

where the detailed expressions of the real functions $W_1(r)$ and $W_2(r)$ are omitted.

It follows from (36) and (39) that

$$\phi_{2rz} = H_{1r} = 0 \quad \text{at } r = 1, z = 0. \tag{41}$$

Then, by time-reducing, (35), (38), (40) and (41), ϕ_2 can be expressed as

$$\phi_2 = \mathrm{i}(B_{0,W_1}(r,z) + B_{m,W_2}(r,z)\cos 2m\theta)(p^2(\tau)\mathrm{e}^{\mathrm{i}\Omega t} - \overline{p}^2(\tau)\mathrm{e}^{-\mathrm{i}\Omega t}). \tag{42}$$

where $B_{0,W_1}(r,z)$ and $B_{m,W_2}(r,z)$ are the solutions of the following equations for $B_{\ell,G}$:

$$\begin{aligned}&B_{rr} + r^{-1}B_r - 4\ell^2 r^{-2}B + B_{zz} = 0 \quad \text{in } 0 \le r < 1, -h < z < 0,\\ &B_z - 4\Omega^2 B + TB_{zzz} = G(r) \quad \text{at } z = 0,\\ &B_r = 0 \quad \text{at } r = 1, \quad B_z = 0 \quad \text{at } z = -h,\\ &B_{rz} = 0 \quad \text{at } r = 1, z = 0.\end{aligned}$$

$B_{\ell,G}(r,z)$ can be found and is given by

$$B_{\ell,G}(r,z) = \sum_{j=1}^{\infty} d_j \cosh(\sigma_{(2\ell)j}(h+z))J_{2\ell}(\sigma_{(2\ell)j}r)$$

where

$$\begin{aligned}&d_j = \int_0^1 G(r)J_{2\ell}(\sigma_{(2\ell)j}r)r\mathrm{d}r(\Delta_{(2\ell)j}||J_{2\ell}(\sigma_{(2\ell)j}r)||^2\cosh(\sigma_{(2\ell)j}h))^{-1},\\ &\Delta_{(2\ell)j} = \Omega^2_{(2\ell)j} - 4\Omega^2 \neq 0,\\ &||J_{2\ell}(\sigma_{(2\ell)j}r)||^2 = \int_0^1 J^2_{2\ell}(\sigma_{(2\ell)j}r)\mathrm{d}r = (\sigma^2_{(2\ell)j} - 4\ell^2)J^2_{2\ell}(\sigma_{(2\ell)j})(\sigma_{(2\ell)j})^{-1},\end{aligned}$$

and we assume that there exists no j such that $J'_{2\ell}(\sigma_{(2\ell)j}) = 0$ and $\Omega_{(2\ell)j} = 2\Omega$ for $\ell = 0$ or m. From (36), at $z = 0$,

$$\eta_{2t} = \phi_{2z} - H_1 = \mathrm{i}(\tilde{W}_1(r) + \tilde{W}_2(r)\cos 2m\theta)(p^2\mathrm{e}^{\mathrm{i}\Omega t} - \overline{p}^2(\tau)\mathrm{e}^{-\mathrm{i}\Omega t}), \tag{43}$$

Where $\tilde{W}_i(r), i = 1$ to 2, are real, and their detailed expressions are not presented here. Assume η_2 has the form of

$$\eta_2 = \hat{\eta}_2\mathrm{e}^{\mathrm{i}\Omega t} + \overline{\hat{\eta}_2}\mathrm{e}^{-\mathrm{i}\Omega t}. \tag{44}$$

We substitute (44) in (43) and obtain

$$\hat{\eta}_2 = (2\Omega)^{-1}(\tilde{W}_1(r) + \tilde{W}_2(r)\cos 2m\theta)p^2(\tau).$$

Hence

$$\eta_2 = (2\Omega)^{-1}(\tilde{W}_1(r) + \tilde{W}_2(r)\cos 2m\theta)p^2(\tau).(p^2\mathrm{e}^{\mathrm{i}\Omega t} - \overline{p}^2(\tau)\mathrm{e}^{-\mathrm{i}\Omega t}) \tag{45}$$

We remark that since a resonance frequency is in general not an eigenfrequency of the time-reduced homogeneous equations corresponding to (35) to (39), a solvability condition cannot be obtained at this stage for a resonance frequency and we have to proceed to the equations for the third-order approximations.

4. Third-order approximations

The equations for the third-order approximations are

$$\Delta_3\phi_3 = 0 \quad \text{in } 0 \le r < 1, -h < z < 0, \tag{46}$$

at $z = 0$:

$$\begin{aligned}
\phi_{3z} - \eta_{3t} &= -\phi_{2zz}\eta_1 - \phi_{1zz}\eta_2 - (1/2)\phi_{1zzz}\eta_1^2 + \eta_{1\tau} + \eta_{2r}\phi_{1r} + \eta_{1r}\phi_{2r} \\
&\quad + \eta_{1r}\eta_1\phi_{1rz} + r^2(\eta_{1\theta}\phi_{2\theta} + \eta_{2\theta}\phi_{1\theta} + \eta_{1\theta}\phi_{1\theta z}\eta_1) \\
&= H_4(\eta_1, \phi_1, \eta_2, \phi_2),
\end{aligned} \tag{47}$$

$$\begin{aligned}
\phi_{3t} - \eta_3 &= -\phi_{2tz}\eta_1 - \phi_{1zt}\eta_2 - (1/2) + \phi_{1zzt}\eta_1^2 - \phi_{1\tau} - \phi_{1r}\phi_{2r} - \phi_{1r}\phi_{1rz}\eta_1 \\
&\quad - r^{-2}(\phi_{1\theta}\phi_{2\theta} + \phi_{1\theta}\phi_{1\theta z}\eta_1) - \phi_{1z}\phi_{2z} - \phi_{1z}\phi_{1zz}\eta_1 \\
&\quad + T(2r^{-3}\eta_{1r}\eta_{1\theta}^2 - 2r^{-2}\eta_{1r}\eta_{1r\theta}\eta_{1\theta} + r^{-2}\eta_{1rr}\eta_{1\theta}^2 + r^{-2}\eta_{1r}^2\eta_{1\theta\theta}^2 \\
&\quad + r^{-1}\eta_{1r}^3 - (3/2)(\eta_{1r}^2 + r^{-2}\eta_{1\theta}^2)(\eta_{1rr} + r^{-2}\eta_{1\theta\theta} + r^{-1}\eta_{1r})) \\
&\quad + T\Delta_2\eta_3 \\
&= H_5(\eta_1, \phi_1, \eta_2, \phi_2) + T\Delta_2\eta_3,
\end{aligned} \tag{48}$$

$$\phi_{3r} = F_t = f(z)\cos m\theta(e^{i(\Omega t + \lambda\tau + \delta)} + e^{-i(\Omega t + \lambda\tau + \delta)}) = \tilde{M} \text{ at } r = 1, \tag{49}$$

$$\phi_{3z} = 0 \quad \text{at } z = -h. \tag{50}$$

At $r = 1$ and $z = 0$, under Evans's edge condition

$$\eta_{3r} = K = (i\omega)^{-1}\alpha\cos m\theta(e^{i(\Omega t + \lambda\tau + \delta)} - e^{-i(\Omega t + \lambda\tau + \delta)}) \tag{51}$$

or under Hocking's edge condition,

$$\eta_{3r} = \hat{\delta}\eta_{1t}. \tag{52}$$

It follows from (47) and (48) that

$$\phi_{3tt} + \phi_{3z} - T\Delta_2\phi_{3z} : H_4 + H_{5t} - T\Delta_2 H_4 = \tilde{N} \quad \text{at } z = 0. \tag{53}$$

By (47), (51) implies

$$\begin{aligned}
\phi_{3rz} &= K_t + H_{4r} = K_t \\
&= \alpha\cos m\theta(e^{i(\Omega t + \lambda\tau + \delta)} + e^{-i(\Omega t + \lambda\tau + \delta)}) \\
&= \tilde{D}_E \quad \text{at } r = 1, z = 0.
\end{aligned} \tag{54}$$

and (52) implies

$$\begin{aligned}
\phi_{3rz} &= \hat{\delta}\phi_{1zt} + H_{4r} = \hat{\delta}\phi_{1zt} \\
&= i\hat{\delta}\Omega\sigma\sinh(\sigma h)J_m(\sigma)\cos m\theta(p(\tau)e^{i\Omega t} - \overline{p}(\tau)e^{-i\Omega t}) \\
&= \tilde{D}_H \quad \text{at } r = 1, z = 0.
\end{aligned} \tag{55}$$

Since only the terms with the factor $\cos m\theta e^{i\Omega t}$ in the nonlinear interactions of $\tilde{M}$, $\tilde{N}$, $\tilde{D}_E$ and $\tilde{D}_H$ can cause resonance, we may express ϕ_3, $\tilde{M}$, $\tilde{N}$, $\tilde{D}_E$ and $\tilde{D}_H$ as

$$\phi_3 = \psi_3(r, z, \tau)\cos m\theta e^{i\Omega t} + \Phi_3(r, z, \theta, t, \tau) + c.c., \tag{56}$$
$$\tilde{M} = M(z, \tau)\cos m\theta e^{i\Omega t} + \hat{M}(z, \theta, t, \tau) + c.c., \tag{57}$$
$$\tilde{N} = N(r, \tau)\cos m\theta e^{i\Omega t} + \hat{N}(r, \theta, t, \tau) + c.c., \tag{58}$$
$$\tilde{D}_E = D_E(\tau)\cos m\theta e^{i\Omega t} + \hat{D}_E(\theta, t, \tau) + c.c. \tag{59}$$
$$\tilde{D}_H = D_H(\tau)\cos m\theta e^{i\Omega t} + \hat{D}_H(\theta, t, \tau) + c.c. \tag{60}$$

where Φ_3, $\hat{M}$, $\hat{N}$, $\hat{D}_E$ and $\hat{D}_H$ do not contain the factor $\cos m\theta e^{i\Omega t}$. Substitution of (56) to (60) in (46) to (55) yields

$$\psi_{3rr} + r^{-1}\psi_{3r} - m^2 r^{-2}\psi_3 + \psi_{3zz} = 0 \quad \text{in } 0 \le r < r, -h < z < 0, \tag{61}$$
$$\psi_{3z} - \Omega^2\psi_3 - T(\psi_{3rrz} + r^{-1}\psi_{3rz} - m^2 r^{-2}\psi_{3z}) = N(r, \tau) \quad \text{at } z = 0, \tag{62}$$
$$\psi_{3r} = M(z, \tau) \quad \text{at } r = 1, \tag{63}$$
$$\psi_{3z} = 0 \quad \text{at } z = -h, \tag{64}$$
$$\psi_{3rz} = D_j(\tau) \quad \text{at } r = 1, z = 0 \tag{65}$$

where $j = E$ for Evans's edge condition and $j = H$ for Hocking's edge condition.

Since Ω is the resonance frequency and $J_m(\sigma r)\cosh(\sigma(h + z))$ is the corresponding eigenfunction for (27) to (30), the extra terms M, N and D_j in (61) to (65) must satisfy a solvability condition. We multiply (61) by $rJ_m(\sigma r)\cosh(\sigma(h + z))$, integrate the resulting equation over V, and make use of the divergence theorem and (62) to (65) to obtain the solvability condition

$$\frac{\cosh(\sigma h)}{1 + T\sigma^2}\int_0^1 N(r, \tau)rJ_m(\sigma r)\mathrm{d}r + \int_{-h}^0 M(z, \tau)\cosh\sigma(h + z)\mathrm{d}z J_m(\sigma) + T\frac{\cosh(\sigma h)}{1 + T\sigma^2}J_m(\sigma)D_j(\tau) = 0 \tag{66}$$

where

$$\begin{aligned} M(z, \tau) &= f(z)e^{i(\lambda\tau+\delta)}, \\ D_j(\tau) &= \alpha e^{i(\lambda\tau+\delta)} \quad \text{if } j = E, \\ D_j(\tau) &= i\hat{\delta}\Omega\sigma\sinh(\sigma h)J_m(\sigma)p(\tau) \quad \text{if } j = H, \\ N(r, \tau) &= iN_1(r)p'(\tau) + N_2(r)p(\tau)|p(\tau)|^2, \\ N_1(r) &= -2\Omega J_m(\sigma r)\cosh(\sigma h), \\ {}'' \, ' \, '' &= \frac{\mathrm{d}}{\mathrm{d}\tau}, \end{aligned}$$

and the detailed expressions of $N_2(\tau)$ are not presented here.

Hence it follows from (66) that for Evans's edge condition, that is, $j = E$,

$$ia_0 p' + a_1 p|p|^2 + a_E e^{i(\lambda\tau+\delta)} = 0, \tag{67}$$

and for Hocking's edge condition, that is, $j = H$,

$$ia_0 p' + a_1 p|p|^2 + a_H e^{i(\lambda\tau+\delta)} + ia_3 p = 0, \tag{68}$$

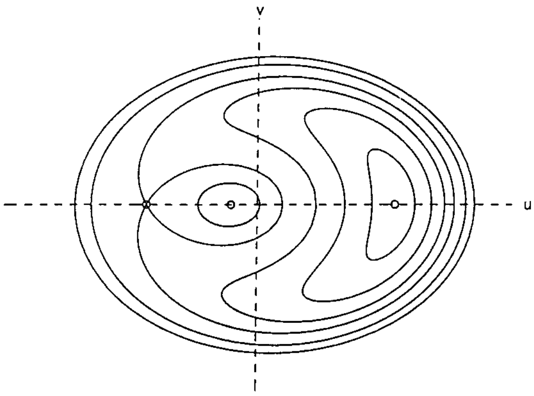

Figure 2. The phase plane of (69), (70) as $c_1 < 0$, $\lambda < 0$, $0 \le c_2 < c_0$.

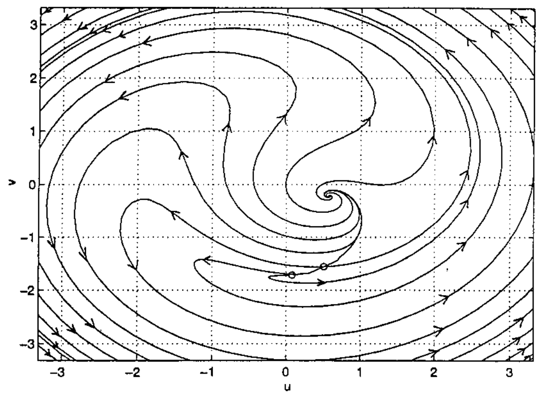

Figure 3. A phase plane of (75), (76).

where

$$
\begin{aligned}
a_0 &= \cosh(\sigma h)(1+T\sigma^2)^{-1}\int_0^1 N_1(r) r J_m(\sigma r)\mathrm{d}r,\\
&= -2\Omega\cosh^2(\sigma h)(1+T\sigma^2)^{-1}||J_m(\sigma r)||^2 < 0,\\
a_1 &= \cosh(\sigma h)(1+T\sigma^2)^{-1}\int_0^1 N_2(r) r J_m(\sigma r)\mathrm{d}r,\\
a_E &= J_m(\sigma)\left(\int_{-h}^0 f(z)\cosh\sigma(h+z)\mathrm{d}z + \alpha T\cosh(\sigma h)(1+T\sigma^2)^{-1}\right),\\
a_H &= J_m(\sigma)\int_{-h}^0 f(z)\cosh\sigma(h+z)\mathrm{d}z,\\
a_3 &= T\hat{\delta}\Omega\sigma\sinh(\sigma h)J_m^2(\sigma)\cosh(\sigma h)(1+T\sigma^2)^{-1} > 0.
\end{aligned}
$$

5. Discussion

In this section, the behavior of the solutions of (67) and (68) is studied. First we consider (67). Let $p(\tau) = (u+iv)\mathrm{e}^{\mathrm{i}(\lambda\tau+\delta)}$ where u and v are real, and substitution of it in (67) yields

$$u' = \lambda v - c_1 v(u^2+v^2) = -H_v, \tag{69}$$
$$v' = -\lambda u + c_1 u(u^2+v^2) + c_2 = H_u, \tag{70}$$

where $c_1 = a_1/a_0$, $c_2 = a_E/a_0$, and

$$H = (c_1/4)(u^2+v^2)^2 - (\lambda/2)(u^2+v^2) + c_2 u \equiv \text{constant}. \tag{71}$$

Since (69) and (70) are a 2-dimensional Hamiltonian system, by (71) all solutions of (69) and (70) are bounded and all trajectories in the phase plane (u,v) are closed and not mutually transversal. The solution behavior depends upon the fixed points of (69) and (70). They are either centers or saddle points. As an example, the typical phase plane of (69) and (70) for

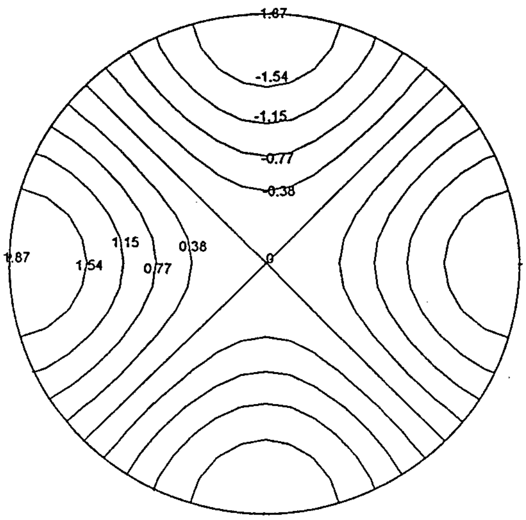

Figure 4. A contour plot of the free surface for $\Omega = 1.7431$.

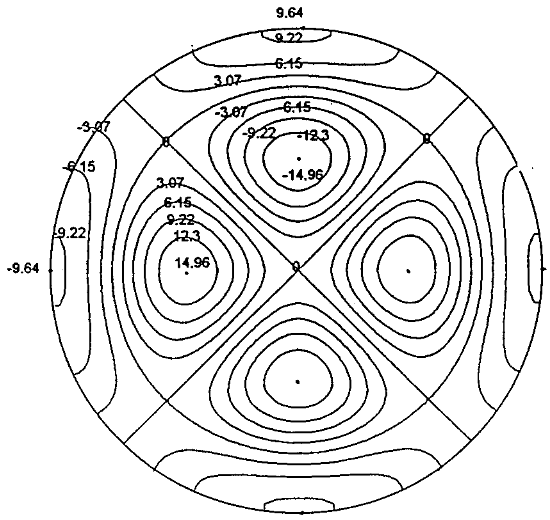

Figure 5. A contour plot of the free surface for $\Omega = 3.1142$.

$c_1 < 0, \lambda < 0, 0 \leq c_2 < c_0 = (2|\lambda|/3)(\lambda/(3c_1))^{1/2}$ is shown in Figure 2, where two fixed points are centers and one fixed point is a saddle. A detailed discussion of (69) and (70) can be found in [11] (Part 3, pp. 162–164).

Next We shall study the behavior of solutions for (68) under Hocking's condition. Let $p(\tau) = \tilde{W}(\tau)e^{i(\lambda\tau+\delta)}$ and (68) becomes

$$i\tilde{W}' = \lambda\tilde{W} - b_1\tilde{W}|\tilde{W}|^2 - b_2 - ib_3\tilde{W}, \tag{72}$$

where $b_1 = a_1/a_0$, $b_2 = a_H/a_0$, $b_3 = a_3/a_0$ and $b_3 < 0$ because $a_0 < 0$ and $a_3 > 0$.

If $b_1 = 0$, then (72) is linear and the solution of (72) is

$$\tilde{W} = -b_2/(b_3 + i\lambda) + Ce^{-(b_3+i\lambda\tau)},$$

where C is a constant. Since $b_3 < 0$, $\tilde{W}$ approaches $-b_2/(b_3 + i\lambda)$ as $\tau \to -\infty$ and $|\tilde{W}|$ goes to ∞ as $\tau \to -\infty$.

If $b_1 \neq 0$, we let $\tilde{W} = kW$ as $b_1 > 0$ or $\tilde{W} = -k\tilde{W}$ as $b_1 < 0$ where $k = |b_1|^{1/2}$. Then (72) becomes

$$iW' = \lambda W - W|W|^2 - b - ib_3W \quad \text{if } b_1 > 0, \tag{73}$$

or

$$iW' = \lambda W - W|W|^2 - b - ib_3W \quad \text{if } b_1 < 0, \tag{74}$$

where $b = b_2/k$. Since (73) and (74) only differ by the sign of λ, we shall just consider (73) for any λ.

Let $W = u + iv$ where u and v are two real functions of τ. Substitution in (73) yields a system of two first order differential equations

$$u' = \lambda v - v(u^2 + v^2) - b_3u = f_1(u, v), \tag{75}$$

$$v' = -\lambda u + u(u^2 + v^2) - b_3v + b = f_2(u, v). \tag{76}$$

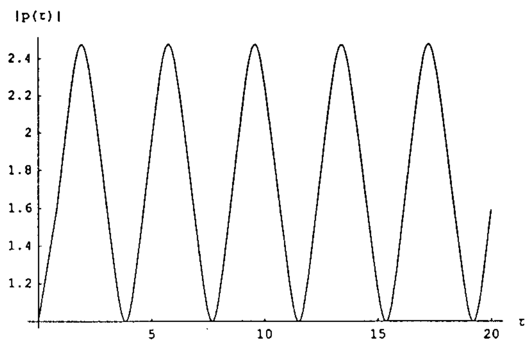

Figure 6. $|p(\tau)|$ vs τ for (69), (70).

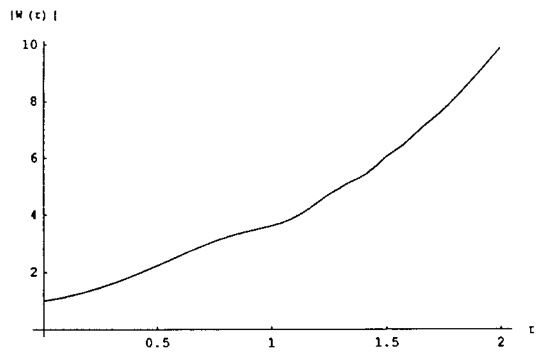

Figure 7. $|W(\tau)|$ vs τ for (75), (76).

The fixed points of (75) and (76) are the solutions (u^*, v^*) satisfying

$$P(v^*) = v^{*3} - 2\lambda b_3 b^{-1} v^{*2} + (\lambda^2 + b_3^2) b_3^2 b^{-2} v^* - b_3^3 b^{-1} = 0,$$

and

$$u^* = b_3^{-1} v^* (\lambda - b b_3^{-1} v^*).$$

In addition,

$$u^{*2} + v^{*2} = b b_3^{-1} v^*.$$

Since $b_3 < 0$, a fixed point of (75) and (76) can be a spiral source, nodal source, or saddle point. In the case of $\lambda \leq 0$, there is only one fixed point, a spiral source. In the case of $\lambda > 0$, the fixed points depend upon the values of R_1, R_2, R_3, and R_4 as functions of b_3 and λ defined below:

$$R_1(b_3, \lambda) = b_3^2 \lambda, \qquad R_3(b_3, \lambda) = \frac{2}{27}(9b_3^2\lambda + \lambda^3 - (\lambda^2 - 3b_3^2)^{3/2}),$$

$$R_2(b_3, \lambda) = \frac{b_3^2}{3}\lambda + \frac{4}{27}\lambda^3, \quad R_4(b_3, \lambda) = \frac{2}{27}(9b_3^2\lambda + \lambda^3 - (\lambda^2 - 3b_3^2)^{3/2}).$$

A typical phase plane of (75) and (76) in the case of $\min\{R_1, R_2\} \geq b^2 > R_3$ and $b^2 \neq R_2$ as $\lambda > -2b_3$ is shown in Figure 3. There are three fixed points, a spiral source, a saddle source and a nodal source. We see that the amplitude of all solutions are unbounded except at the fixed points and along the orbits which connect two fixed points. The same behavior of the solutions of (75) and (76) is also displayed in the phase planes for other cases. In fact, a rigorous proof based on Poincare-Bendixson theorem and Bendixson criterion (Chapter 1, [12]) can be constructed to show that there are no closed orbits for (75) and (76). A detailed discussion of (75) and (76) and the proof can also be found in [11] (Part 3, pp. 165–171).

To illustrate the results obtained, we shall study (34) numerically, which gives the expression for the first-order approximation η_1 to the free surface. Figures 4 and 5 present the contour plots of η_1 for constant t and τ, where we choose m=2, h=0.5 and T=0.01. In Figure 4, $\Omega = 1.7431$ is the first resonance frequency and four extremum points appear at the rim. In

Figure 5 $\Omega = 3.1142$ is the second resonance frequency and the extremum points also appear inside the rim. These graphs are similar to those for the linear problem considered in [4]. Then we plot $\sqrt{(u^2 + v^2)}$ vs τ for typical cases under Evans's and Hocking's condition. In Figure 6 we choose $\lambda = 3$, $c_1 = 1$, $c_2 = 1$, u(0) $= -1$ and $v(0) = 0$, and the graph represents a bounded and periodic solution for (69), (70) under Evans's condition. In Figure 7 we choose $\lambda = 3$, $b = 1.72$, $b_3 = -1$, u(0) $= -1$ and $v(0) = 0$, and the graph represents an increasing solution with time for (75), (76) under Hocking's condition.

Under Evans's condition resonance frequencies can be obtained by setting $\tilde{K}$ equal to zero and linearizing the governing equations [3]. However, under Hocking's condition it is shown [6] that there are no real resonance frequencies of the linearized governing equations without forcing. In the theory developed here, we assume $\tilde{\delta}$ in Hocking's condition is small so that the equations for the first-order approximation ϕ_1 yield the resonance frequencies under the classical edge condition. The two amplitude equations derived under these two edge conditions also display quite different behavior. The wave amplitude is bounded for any time under Evans's condition, but generally increases with time under Hocking's condition. As indicated in [1], [2], the amplitude of an excited wave increases with time, and eventually several jets spurts high into the air, depending on the setting of the experiments. At this stage we can only conclude that an edge condition at a contact line has a dominant effect on the behavior of an excited wave, and needs more investigation.

References

1. Qing-Ding Wei, Dajun Wang, Ben Yan Xiang-Dong Du, and Ju Chen, A visualization study on water spray of Dragon Washbasin. *Atlas of Visualization III*, The Visualization Society of Japan. New York: CRC Press at Boca Raton (1997) pp. 169–179.
2. D. Y. Hsieh and P. Denissenko, Water waves in a circular elastic vessel: The experiment. Technical Report, Department of Mathematics, The Hong Kong University of Science & Technology (1998) 24 pp.
3. D. Henderson and J. W. Miles, Surface wave damping in a circular cylinder with a fixed contact line. *J. Fluid Mech.* (1994) 285–299.
4. M. C. Shen, S. M. Sun and D. Y. Hsieh, Forced capillary-gravity waves in a circular basin. *Wave Motion* 18 (1993) 401–412.
5. D. V. Evans, The effect of surface tension on the waves produced by a heaving circular cylinder. *Proc. Cambridge Phil. Soc.* 64 (1968) 833–847.
6. M. C. Shen and N. S. Yeh, Exact solution for forced capillary-gravity waves in a circular basin under Hocking's edge condition. *Wave Motion* 26 (1997) 117–126.
7. L. M. Hocking, The damping of capillary-gravity waves at a rigid boundary. *J. Fluid Mech.* 179 (1987) 253–266.
8. J. Miles, The capillary boundary layer for standing waves. *J. Fluid Mech.* 222 (1991) 197–205.
9. J. Miles, On forced capillary-gravity waves in a circular cylinder. *Wave Motion* 23 (1996) 387–389.
10. S. M. Sun, M. C. Shen and D. Y. Hsieh, Nonlinear forced capillary-gravity waves in a circular basin. *Wave Motion* 21 (1995) 331–341.
11. C. C. Chang, Nonlinear theories of forced surface waves in a circular basin, Ph.D. thesis, Department of Mathematics, University of Wisconsin (1999) 205 pp.
12. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* New York: Springer-Verlag (1983) 453 pp.